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Symbol and Bit Error Probability for Coded TQAM in AWGN Channel

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Abstract

The performance of coded modulation scheme based on the application of integer codes to triangular quadrature amplitude modulation (TQAM) with 2^{2m} points constellation is investigated. A method of calculating the exact value of symbol error rate over AWGN channel in the case of TQAM combined with encoding by integer codes is described. Computation of error probability in the coded case is much more difficult because of more complex forms and in a sense uncertainties of decision regions. It is demonstrated that in the coded case computer simulation is a very good alternative. The results of simulations (symbol and bit error probabilities) in the case of coded 16, 64, and 256-TQAM simulations are graphically presented.

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Keywords: TQAM, Integer codes, BER, SER, simulations

1 Introduction

The term coded modulation denotes a scheme that integrates coding and modulation techniques. In modern digital communication systems, high-order modulation constellations are preferred for high-speed data transmission. One of the most used modulations in commercial communication systems is square quadrature amplitude modulation (SQAM). SQAM scheme with its simple detection procedure is easy for implementation and demonstrates a good performance.

Recently, the so called triangular quadrature amplitude modulation (TQAM) have been studied more carefully. The signal points of TQAM constellation are vertexes of a lattice of equilateral triangles and the constellation is symmetric with respect to the origin. The comparison of TQAM with SQAM given in [8] shows that the former is more power efficient while preserves in general the low detection complexity of the latter. In [9] a formula for calculating the average energy per symbol of the TQAM is derived as well as approximate values of symbol error rate (SER) and bit error rate (BER) of the TQAM in the presence of additive white Gaussian noise (AWGN) are given. Usually convolution codes are used to correct erroneously detected signal points but another approach is to use so called integer codes. They

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are codes defined over finite rings of integers. These codes enable lower complexity decoding and in contrast to the traditional block codes they are designed to correct errors of a given type. It means that for a given channel and modulator we can construct integer code capable to correct the type of errors, which are the most common for this channel.

An upper and a lower bounds for symbol error probability (SER) in the case of AWGN channel, both in uncoded and coded with integer codes case, are derived in [2]. The exact values of SER in uncoded case are computed in [7].

In this work we shall investigate the performance of coded modulation scheme based on the application of integer codes to TQAM constellation with 2^{2m} points. Necessary definitions and results are given in Section 2. In Section 3 we describe how to compute the exact value of SER in the case of TQAM over AWGN channel coded by integer codes. Section 4 describes the simulations that have been carried out.

2 Preliminaries

2.1 TQAM constellation

In this paper we consider TQAM constellation of $M = 2^{2m}$ signal points placed in $L = 2^m$ rows parallel to the horizontal axis with L signal point in each row. The signal points form a lattice of equilateral triangles with side equals $2d$ and the constellation is symmetric with respect to the origin. The distance between horizontal rows is $h = d\sqrt{3}$ (which is the altitude of the of equilateral triangles). The average energy per signal point E_s and the parameter d are connected by the equation ([9])

$$d^2 = \frac{12}{7L^2 - 4} \cdot E_s. \tag{1}$$

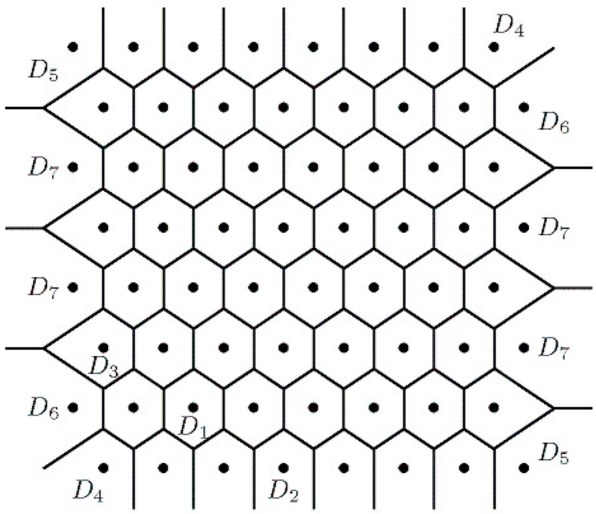

Figure 1: 64-TQAM constellation - uncoded case

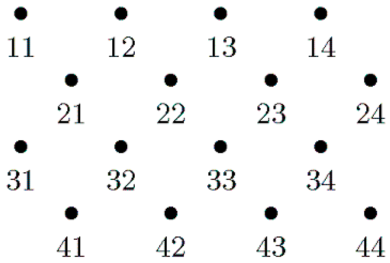

Figure 2: Coded 16-TQAM

Usually $E_s = 1$ is assumed.

The dominated form of detection regions is hexagon with side equals $a = 2d/\sqrt{3}$ (the regions of contour points are more complex). The 64-TQAM constellation with the different type of detection regions in uncoded case is presented in Figure 1

The power gain of M-ary TQAM over M-ary SQAM in decibels, as it is given in [9], is

$$10\log_{10}\left(\frac{8M-8}{7M-4}\right) \xrightarrow[M\to\infty]{} 0.5799 \text{ dB}$$

For $M = 16,\ 64,\ 256$ the power gain is 0.458, 0.5505 and 0.5726, respectively.

We consider an AWGN channel with power spectral density N_0 watts/hertz. Hence if a signal point $\mathbf{s}$ is sent through the channel the received signal $\mathbf{r} = \mathbf{s} + \mathbf{n}$, where $\mathbf{n}$ is two dimensional zero-mean Gaussian random process with variance $\sigma^2 = N_0/2$. At the other end of the channel the detector estimates $\mathbf{r}$ and has to decide which signal point has been sent.

In this paper we use the error function to express the probability of normal distribution (the case of AWGN):

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt; \qquad \Pr\{\,0 \le x \le d\,\} = \frac{1}{2}\mathrm{erf}\left(\frac{d}{\sigma\sqrt{2}}\right)$$

(see [1])

2.2 Integer codes

Herein we give some necessary definitions and notations which we shall use in the next sections. More details the reader can find in the mentioned papers.

Integer codes were proposed by Varshamov and Tenengolz [10] in 1965 for correcting single insertion/deletion per codeword, but in [4, 5] it was demonstrated that such classes of codes are very suitable for realization of coded modulation procedures. The applications of integer codes with different modulation schemes, in partial with SQAM, are discussed also in [3, 6].

Definition 2.1. [5] Let $\mathbf{Z}_A$ be the ring of integers modulo A. An *integer code* of length n with parity-check matrix $H \in \mathbf{Z}_A^{m\times n}$, is referred to as a subset of $\mathbf{Z}_A^n$, defined by

$$\mathbf{C}(H,\mathbf{d}) = \{\mathbf{c} \in \mathbf{Z}_A^n \,|\, \mathbf{c}H^T = \mathbf{d} \bmod A\}$$

where $\mathbf{d} \in \mathbf{Z}_A^m$.

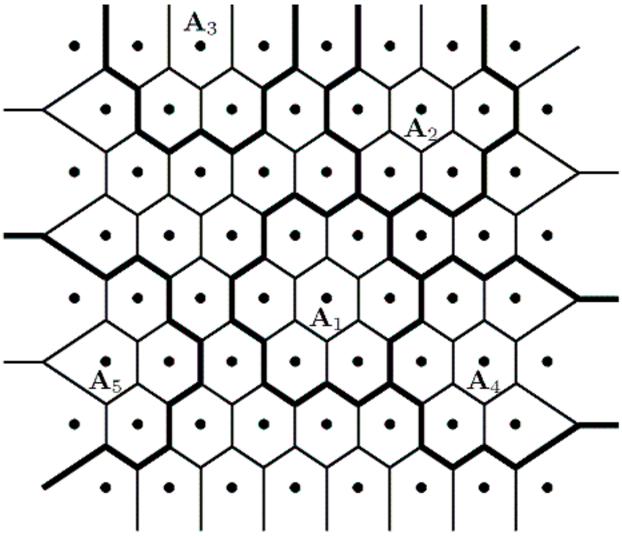

Figure 3: Several error detection regions in the case of coded 64-TQAM constellation

Assume that a signal point s_i of a given signal constellation is sent through an AWGN-channel. At the other end the detector estimates the received signal r_i and gives signal point s_j at the output. If $j \neq i$ the detector has taken a wrong decision. In terms of block codes over $\mathbf{Z}_A$ the aforesaid can be described in the following way. When a codeword $\mathbf{c} \in \mathbf{C}(\mathbf{w}, d)$ is sent through a communication channel (usually noisy) the received vector can be written in the form

$$\mathbf{r} = \mathbf{c} + \mathbf{e},$$

where $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbf{Z}_A^n$ denotes the error vector. It is clear that the different signal points have not the same chance to be a result of decision process. The probability signal point s_j to appear at the output of the detector depends on the Euclidean distance between s_j and really-sent signal point s_i. In terms of codes over $\mathbf{Z}_A$ it means that the elements of $\mathbf{Z}_A$ are not equally probable as a value taken by e_i. Which elements of $\mathbf{Z}_A$ are more probable depends on the chosen indexing of the signal points by the elements of $\mathbf{Z}_A$. Therefore, it makes sense to consider (there is a point in considering) the next definition.

Definition 2.2 [5] The code $\mathbf{C}(H, \mathbf{d})$ is a t-multiple $(\pm e_1, \pm e_2, \ldots, \pm e_s)$-error correctable if it can correct up to t errors with values from the set $\{\pm e_1, \pm e_2, \ldots, \pm e_s\}$ which are occurred in a codeword.

Remark: Without loss of generality in the definitions above we can assume that $\mathbf{d} = \mathbf{0}$. For convenience of a notation we shall use $\mathbf{C}$ instead of $\mathbf{C}(H, \mathbf{0})$.

To decode integer codes one can use a hard decoding algorithm [6]. This algorithm uses a look-up table which maps each syndrome value to the corresponding error vector. So, the complexity of the algorithm is linear with respect to the alphabet size A.

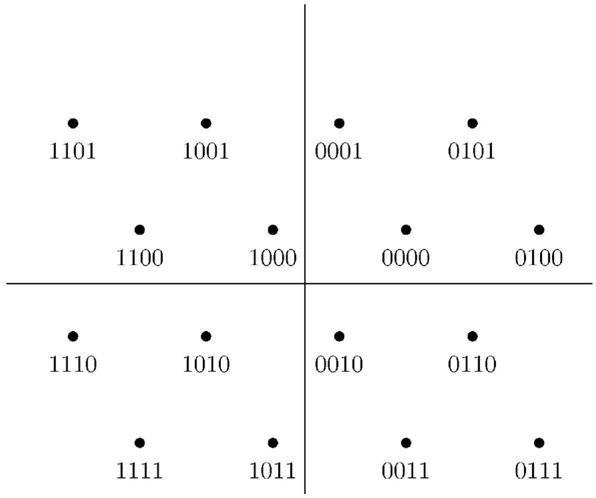

Figure 4: 16-TQAM constellation with Grey mapping

3 SER in coded case

We analyze 2^{2m}-TQAM modulations combined with integer error correcting codes. In the case of 16-TQAM we use two codes over $\mathbb{Z}_5$ with a generator matrix $\mathbf{H} = (1, 2)$ and number the points by pair of integers as it is shown in Figure 2: the first integer shows the horizontal row, and the second number - the vertical see-saw line. This code is a single ± 1-error correcting code. Thus it can correct any case when the sent point is wrongly detected as one of its neighbor (adjacent) points. In the case of 64-TQAM and 256-TQAM the code with the same matrix $\mathbf{H} = (1, 2)$ but over $\mathbb{Z}_9$ and $\mathbb{Z}_{17}$ respectively, can be used.

The encoding procedure is the following. If we have to send the signal point (i, j) we sent two points: (i, j) and (ai, aj) (multiplication is in $\mathbb{Z}_A$, i.e, modulo $A = 5,\ 9$ or 17), where $a = -1/2$ in $\mathbb{Z}_A$. The pair (i, ai) and (j, aj) are codewords, thus the decoder can correct an error of value ± 1 in one of coordinate of each codeword.

In the coded case there are more number of different types of regions of correct detection which are larger and more complex. In the Figure 3 five of the regions are depicted.

Let Q be the average (average on all regions $\mathbf{A_i}$) probability of correct detection and q be the average probability for the uncoded case (average on all regions $\mathbf{D_i}$). Since the code is single error correcting the probability of correct decoding a codeword is $2qQ$, thus the probability of error decoding of a symbol is

$$SER = \sqrt{1 - 2qQ}.$$

(q is computed in [7])

We demonstrate the method of computing the probabilities Q_i by determining the probability of correct detection Q_1 for the simplest, but the dominating region $\mathbf{A_1}$. Due to the symmetry of Gaussian distribution $\mathbf{A_1}$ can be considered as an union of 12 congruent subregion (which are triangles).

Hence the probability Q_1 of correct detection in the case of $\mathbf{A_1}$ is

$$\begin{aligned} \frac{1}{12}Q_1 &= \Pr\left\{0 \le x \le 2d,\ 0 \le y \le \frac{x}{\sqrt{3}}\right\} \\ &+ \Pr\left\{2d \le x \le 3d,\ 0 \le y \le \frac{4d-x}{\sqrt{3}}\right\} \end{aligned}$$

Therefore

$$\begin{aligned} Q_1 &= \frac{6}{\sqrt{\pi N_0}} \int_0^{2d} e^{-\frac{x^2}{N_0}} \operatorname{erf}\left(\frac{x}{\sqrt{3N_0}}\right) dx \\ &+ \frac{6}{\sqrt{\pi N_0}} \int_{2d}^{3d} e^{-\frac{x^2}{N_0}} \operatorname{erf}\left(\frac{4d-x}{\sqrt{3N_0}}\right) dx \end{aligned}$$

Since $E_s = 1$ the signal/noise ratio

$$SNR_s = \frac{E_s}{N_0} = \frac{1}{N_0}.$$

There are no principle difficulties to compute Q but the expressions are too complex, and thus, not practical. The attempt to follow the same approach for BER will result in even more complex expressions. Hence, we consider simulations of the described communication procedure as a better approach. The availability of software for simulation enables more flexibility for investigating the influence on the performance of chosen mapping and codes as well as for changing the type of the channel.

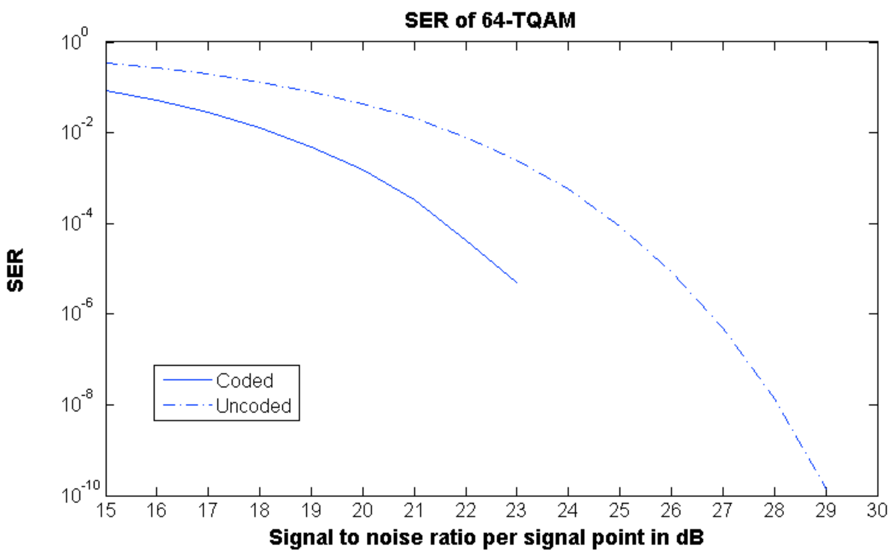

Figure 5: SER for coded and uncoded 64-TQAM

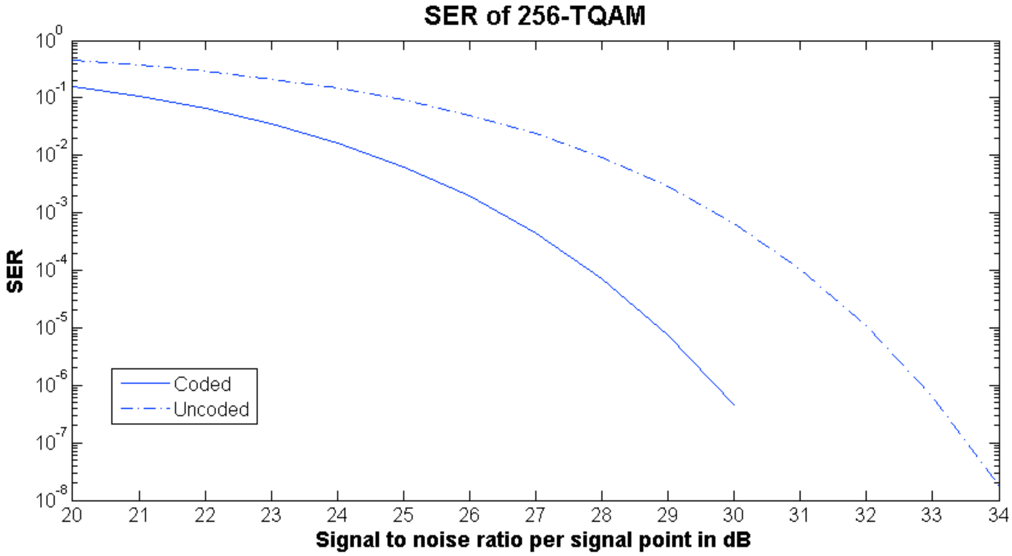

Figure 6: SER for coded and uncoded 256-TQAM

4 Computer Simulations

We have developed a software that simulates communication in AWGN channel based on 16, 64 and 256-TQAMs in two variants: combined with the described in the previous section integer codes with $\mathbf{H} = (1, 2)$ and without any coding. The chosen correspondence between signal points and frame of 4, 6 or 8 bits does not affect decoding procedure and SER, but can increase or decrease BER. It is not difficult to realize (guess) that the best results in uncoded case can be obtained by using so called *Grey mapping* which is illustrate for 16-TQAM in Figure 4. In this case the bit frame of any signal point differs only in one bit for four of the six closest points and only in two bits for the others 2 points.

Another way of mapping between signal points of 2^{2m}-TQAM and binary $2m$-tuples is to use the correspondence between signal points and codewords. If the codeword ab corresponds to a signal point s then $\bar{a}\bar{b}$ is the corresponding binary $2m$-tuple, where $\bar{a}$ and $\bar{b}$ are the binary representations of a and b with the stipulation that 2^m is mapped to all-zeros m-tuple. For example, the point numbered by 14 in Figure 2 corresponds to 0100. This correspondence we refer to as "*standard*"

The Grey mapping does not decrease essentially BER in coded case (and simulations confirm this fact) since in the most of cases the closets points belong to the region of right decoding. Hence we prefer to use "standard" mapping in the coded case.

The software generalizes a random sequence of bits, transforms it into a sequence of signal points according to the chosen mapping and adds to them AWGN corresponding to the prescribed (input) SNR. Then carries out the procedure of detection (and decoding in the coded case) and compare the obtained points with the sent ones to calculate the SER. Also, the software transforms the sequence of detected (decoding) points into a sequence of bits and calculates the BER. Three modes of simulation are realized:

- 'St' - uncoded case with "standard" mapping;

- 'grey' - uncoded case with Grey mapping;
- 'intc' - use of integer codes with $\mathbf{H} = (1, 2)$ and "standard" mapping.

We carried out the simulations by generating pseudo-random sequences of length $\approx 10^{12}$ bits. Then we divide it in blocks of $2m$ bits and transform each block in element of $\mathbb{Z}_{2m+1} \setminus \{0\}$. Two consecutive numbers in the new sequence correspond to one signal point (see Figure 2). Then we encode each coordinate of the pair and produce a new (additional) signal point which is sent on the channel, too. After adding noise we detect which signal point is sent (the decision can be wrong) and carry out the described procedure in reverse order. Then compare the obtained sequences of elements of $\mathbb{Z}_{2m+1}$ and sequences of bits with the original ones to compute SER and BER. The next figures demonstrate the obtained results. Figure 5 and Figure 6 compare SER in uncoded and coded case for 64-TQAM and 256-TQAM, respectively.

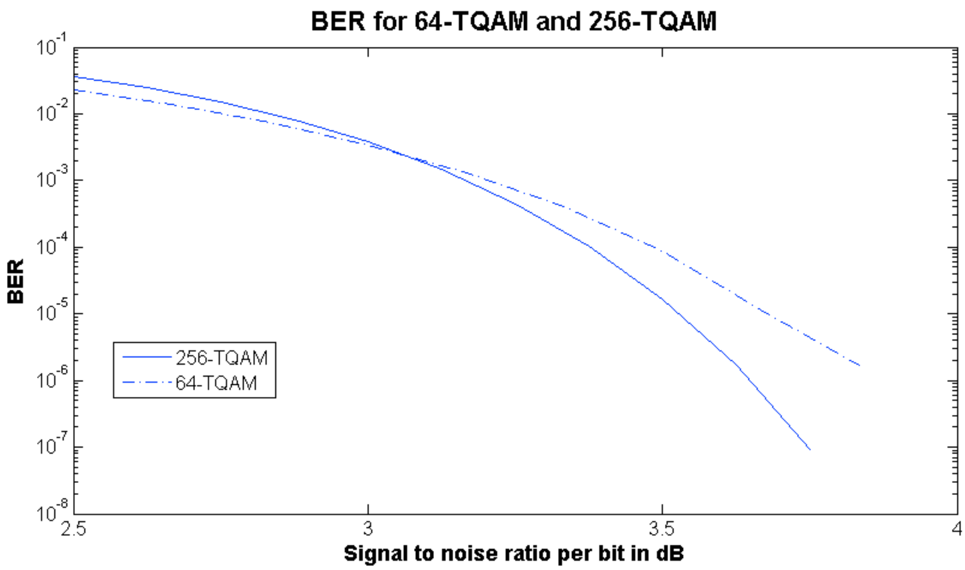

Figure 7: BER for coded 64-TQAM

The break of the curves shows that the probability of error drops down to zero (or less than 10^{-12}) when SNR (in dB) reaches a given value (threshold). The existence of such a threshold can be observed in all cases of coded by integer codes modulation. It is due to the presence of double errors in a codeword (the code is single error correctable) when SNR is just less than this threshold. For SNR greater than the threshold there are no double errors.

Figure 7 presents the BER versus SNR in dB but per bit in the case of coded 64-TQAM and 256-TQAM. For larger values of SNR per bit the BER drops down to zero.

5 Conclusion

We have developed and describe a method for accomplishing coded TQAM and show how the SER in this case can be determined. Also we developed a software that simulates the described procedure in the case of AWGN channel. Indeed the type of the channel as well as codes and mappings can be easily changed. This flexibility enables a wide range of experiments to be carried out. Our further research will be in that direction.

5.1 Acknowledgments

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